AP Calculus AB
3.5: P. 140 #1-10 all, 12-22 even, 25-33 odd

① $y = 1 + x - \cos x$ $\frac{dy}{dx} = 1 + \sin x$	② $y = 2\sin x - \tan x$ $\frac{dy}{dx} = 2\cos x - \sec^2 x$
③ $y = \frac{1}{x} + 5\sin x$ $y = x^{-1} + 5\sin x$ $\frac{dy}{dx} = -x^{-2} + 5\cos x$ $\frac{dy}{dx} = 5\cos x - \frac{1}{x^2}$	④ $y = x\sec x$ $\frac{dy}{dx} = x(\sec x\tan x) + (\sec x)(1)$ $\frac{dy}{dx} = \sec x(x\tan x + 1)$
⑤ $y = 4 - x^2\sin x$ $\frac{dy}{dx} = 0 - [x^2\cos x + (\sin x)(2x)]$ $\frac{dy}{dx} = -x[x\cos x + 2\sin x]$	⑥ $y = 3x + x\tan x$ $\frac{dy}{dx} = 3 + [x\sec^2 x + (\tan x)(1)]$ $\frac{dy}{dx} = 3 + x\sec^2 x + \tan x$
⑦ $y = \frac{4}{\cos x}$ $y = 4\sec x$ $\frac{dy}{dx} = 4\sec x\tan x$	⑧ $y = \frac{x}{1+\cos x}$ $\frac{dy}{dx} = \frac{(1+\cos x)(1) - (x)(-\sin x)}{(1+\cos x)^2}$ $\frac{dy}{dx} = \frac{1 + \cos x + x\sin x}{(1+\cos x)^2}$

(9) $y = \dfrac{\cot x}{1+\cot x}$

$$\frac{dy}{dx} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$$

$$\frac{dy}{dx} = \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1+\cot x)^2}$$

$$\frac{dy}{dx} = -\frac{\csc^2 x}{(1+\cot x)^2} = -\frac{\csc^2 x}{(1+\cot x)^2} \cdot \frac{\sin^2 x}{\sin^2 x}$$

$$\frac{dy}{dx} = -\frac{1}{[\sin x(1+\cot x)]^2} = \boxed{-\frac{1}{(\sin x + \cos x)^2}}$$

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(10) $y = \dfrac{\cos x}{1+\sin x}$

$$\frac{dy}{dx} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$

$$\frac{dy}{dx} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$\frac{dy}{dx} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$\frac{dy}{dx} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$\boxed{\frac{dy}{dx} = -\frac{1}{1+\sin x}}$$

(12) $y = \frac{\tan x}{x}$ normal line at $x = \frac{\pi}{4}$ $\left(\frac{\pi}{4}, \frac{4}{\pi}\right)$

$$\frac{dy}{dx} = \frac{x\sec^2 x - \tan x}{x^2}$$

at $x = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{\frac{\pi}{4}(\sqrt{2})^2 - 1}{\left(\frac{\pi}{4}\right)^2} = \frac{\frac{\pi}{4}(2) - 1}{\frac{\pi^2}{16}} = \frac{\frac{\pi - 2}{2}}{\frac{\pi^2}{16}} = \frac{8(\pi-2)}{\pi^2}$

Slope of normal line: $m = -\left[\frac{\pi^2}{8(\pi-2)}\right] = \frac{\pi^2}{8(2-\pi)}$

$$y = mx + b$$

$$\frac{4}{\pi} = \frac{\pi^2}{8(2-\pi)}\left(\frac{\pi}{4}\right) + b$$

$$\frac{4}{\pi} = \frac{\pi^3}{32(2-\pi)} + b$$

$$b = \frac{4}{\pi} - \frac{\pi^3}{32(2-\pi)}$$

$$\boxed{y = \frac{\pi^2}{8(2-\pi)}x + \frac{4}{\pi} - \frac{\pi^3}{32(2-\pi)}}$$

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(14) $\frac{d}{dx}[\cos x] = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$

$$= \lim_{h\to 0} \frac{\cos x\cos h - \cos x - \sin x\sin h}{h} = \lim_{h\to 0} \frac{\cos x[\cos h - 1] - \sin x\sin h}{h}$$

$$= \lim_{h\to 0} \frac{\cos x[\cos h - 1]}{h} - \lim_{h\to 0} \frac{\sin x\sin h}{h}$$

$$= \left[\lim_{h\to 0}\cos x\right]\left[\lim_{h\to 0}\frac{\cos h - 1}{h}\right] - \left[\lim_{h\to 0}\sin x\right]\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= [\cos x][0] - [\sin x][1] = 0 - \sin x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

(16) a) $\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{(\sin x)\cdot\frac{d}{dx}[\cos x] - (\cos x)\cdot\frac{d}{dx}[\sin x]}{\sin^2 x}$

$= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$

b) $\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)\cdot\frac{d}{dx}[1] - (1)\cdot\frac{d}{dx}[\sin x]}{\sin^2 x}$

$= \frac{(\sin x)(0) - \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}$

$= -\csc x \cot x$

★ I showed $\frac{d}{dx}[\csc x] \longrightarrow$ we did $\frac{d}{dx}[\sec x]$ in class

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(18) $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

To find horizontal tangents:

$\sec^2 x = 0$

There are no values of x for which $\sec^2 x = 0$, so $y = \tan x$ has no horizontal tangents

$y = \cot x$

$\frac{dy}{dx} = -\csc^2 x$

$-\csc^2 x = 0$

There are no values of x for which $-\csc^2 x = 0$, so $y = \cot x$ has no horizontal tangents

(20) $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\frac{dy}{dx} = \sec^2 x$

Parallel to $y = 2x$

$\sec^2 x = 2$

$\sec x = \pm\sqrt{2}$

$x = -\frac{\pi}{4}, \frac{\pi}{4}$

$\boxed{\left(-\frac{\pi}{4}, -1\right) \quad \left(\frac{\pi}{4}, 1\right)}$

(22) $y = 1 + \sqrt{2}\csc x + \cot x$

$$\frac{dy}{dx} = -\sqrt{2}\csc x \cot x - \csc^2 x$$

a) tangent at $(\frac{\pi}{4}, 4)$

$$\frac{dy}{dx} = -\sqrt{2}\csc\frac{\pi}{4}\cot\frac{\pi}{4} - \csc^2\frac{\pi}{4}$$

$$\frac{dy}{dx} = -\sqrt{2}(\sqrt{2})(1) - (\sqrt{2})^2$$

$$\frac{dy}{dx} = -2 - 2 = -4$$

$y = mx + b$

$4 = -4(\frac{\pi}{4}) + b$

$b = 4 + \pi$

$$\boxed{y = -4x + \pi + 4}$$

b) $-\sqrt{2}\csc x \cot x - \csc^2 x = 0$

$-\csc x(\sqrt{2}\cot x + \csc x) = 0$

$-\csc x = 0$ or $\sqrt{2}\cot x + \csc x = 0$

no solutions

$\sqrt{2}\cot x = -\csc x$

$$\frac{\sqrt{2}\cos x}{\sin x} = -\frac{1}{\sin x}$$

$$\cos x = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$x = \frac{3\pi}{4}$$

At $x = \frac{3\pi}{4}$, $y = 1 + \sqrt{2}\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}$

$y = 1 + \sqrt{2}(\sqrt{2}) + (-1) = 2$

Horizontal line through $(\frac{3\pi}{4}, 2)$: $\boxed{y = 2}$

(25) a) limit is $\frac{\pi}{180}$ (based on conversion, degrees $\rightarrow$ radians)

b) 0

c) In degrees, $\lim_{h\to 0} \frac{\sin h}{h} = \frac{\pi}{180}$, $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

$$\frac{d}{dx}\left[\sin x\right] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{x}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \left[\lim_{h\to 0} \sin x\right]\left[\lim_{h\to 0} \frac{\cos h - 1}{h}\right] + \left[\lim_{h\to 0} \cos x\right]\left[\lim_{h\to 0} \frac{\sin h}{h}\right]$$

$$= (\sin x)(0) + (\cos x)\left(\frac{\pi}{180}\right) = \frac{\pi}{180}\cos x$$

d) $$\frac{d}{dx}\left[\cos x\right] = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \cos x - \sin x \sin h}{h} = \lim_{h\to 0} \frac{\cos x[\cos h - 1] - \sin x \sin h}{h}$$

$$= \left[\lim_{h\to 0} \cos x\right]\left[\lim_{h\to 0} \frac{\cos h - 1}{h}\right] - \left[\lim_{h\to 0} \sin x\right]\left[\lim_{h\to 0} \frac{\sin h}{h}\right]$$

$$= (\cos x)(0) - (\sin x)\left(\frac{\pi}{180}\right) = \frac{-\pi}{180}\sin x$$

e) $$\frac{d^2}{dx^2}\left[\sin x\right] = \frac{d}{dx}\left[\frac{\pi}{180}\cos x\right] = \frac{\pi}{180}\left[\frac{-\pi}{180}\sin x\right] = -\frac{\pi^2}{180^2}\sin x$$

$$\frac{d^3}{dx^3}\left[\sin x\right] = \frac{d}{dx}\left[-\frac{\pi^2}{180^2}\sin x\right] = -\frac{\pi^2}{180^2}\left[\frac{\pi}{180}\cos x\right] = -\frac{\pi^3}{180^3}\cos x$$

$$\frac{d^2}{dx^2}\left[\cos x\right] = \frac{d}{dx}\left[-\frac{\pi}{180}\sin x\right] = -\frac{\pi}{180}\left[\frac{\pi}{180}\cos x\right] = -\frac{\pi^2}{180^2}\cos x$$

$$\frac{d^3}{dx^3}\left[\cos x\right] = \frac{d}{dx}\left[-\frac{\pi^2}{180^2}\cos x\right] = -\frac{\pi^2}{180^2}\left[-\frac{\pi}{180}\sin x\right] = \frac{\pi^3}{180^3}\sin x$$

(27) $y = \theta \tan\theta$

$y' = \theta \sec^2\theta + \tan\theta$

$y'' = \theta[2\sec\theta \cdot \sec\theta\tan\theta] + \sec^2\theta + \sec^2\theta$

$y'' = 2\theta\sec^2\theta\tan\theta + 2\sec^2\theta$

$y'' = 2\sec^2\theta[1 + \theta\tan\theta]$

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(29) $\frac{d^{999}}{dx^{999}}[\cos x]$

$\frac{d}{dx}[\cos x] = -\sin x = \frac{d^5}{dx^5}[\cos x]$ Observe the pattern!

$\frac{d^2}{dx^2}[\cos x] = -\cos x = \frac{d^6}{dx^6}[\cos x]$

$\frac{d^3}{dx^3}[\cos x] = \sin x = \frac{d^7}{dx^7}[\cos x]$

$\boxed{\frac{d^{999}}{dx^{999}}[\cos x] = \sin x}$

$\frac{d^4}{dx^4}[\cos x] = \cos x = \frac{d^8}{dx^8}[\cos x]$

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(31) $y = \sin x$

$\frac{dy}{dx} = \cos x$

at $(0,0)$ $\frac{dy}{dx} = 1$

$\boxed{y = x}$

(33) $\frac{d}{dx}[\sin 2x]$

$= \frac{d}{dx}[2\sin x\cos x]$

$= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]$

$= 2[\cos^2 x - \sin^2 x] = 2\cos 2x$